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Resonance Trapping in Dissipative and Antidissipative Systems: An Ergodic Approach

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We propose a weak definition for a resonance trapping in oscillating systems. This definition requires the convergence of orbits, in the sense of measures convergence, to an ergodic invariant measure, supported in a small neighborhood of the resonance zone. Then we apply this definition to a simplified, single-frequency oscillating system which admits a finite number of resonance points. It turns out that, under some assumptions, this generalized concept of resonance trapping may include the case where all resonances are repelling in the classical sense. The analysis is reduced to the investigation of the integrability of the logarithmic singularity with respect to an invariant measure of a reduced mapping.

KEY WORDS: Ergodic measures; resonance trapping; circle maps.

1. INTRODUCTION

In general, resonant trapping in an oscillating nonlinear system of ordinary differential equations is associated with the dissipative nature of the system. The analysis is essentially local (resonance capture) and is concentrated in a neighborhood of a given resonance (see, e.g., refs. 4–7).

In this paper we suggest a different approach which is global in nature, and introduce a definition for a resonant trapping in a generalized sense.

Consider the model system

$$\begin{aligned} \dot{x} &= \varepsilon g(x, \theta, s) \\ \dot{\theta} &= \omega(x) \qquad\qquad (1.1_\varepsilon) \\ \dot{s} &= 1 \end{aligned}$$

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where g is smooth, periodic of period 1 in (x, θ, s) and ω is 1-periodic in x. Thus, we can view (1.1_ε) as a smooth dynamical system on the torus $\mathbb{T}^3$. For convenience we will denote $\mathbb{T}^3$ as $\mathbb{S} \otimes \mathbb{T}^2$, where the unit circle $\mathbb{S}$ is the phase space for the slow variable $\{x\}$ and $\mathbb{T}^2$ is the phase space for the fast variables $\{\theta, s\}$.

A point $x \in \mathbb{S}$ is called a resonant point of (1.1_ε) if $\exists k, l$ relatively prime integers such that (a) $k\omega(x) + l = 0$; and (b) there exists an integer $n \neq 0$ for which $\mathscr{g}^{(n)}_{(k,l)}(x)$ is not identically zero, where

$$\mathscr{g}^{(n)}_{(k,l)}(x) := \int_0^1 \int_0^1 g(x, \theta, s) e^{2\pi i n(k\theta + ls)}\, d\theta\, ds$$

In this paper we refer to the problem of resonant trapping in a simplified version of (1.1_ε) from an ergodic theory point of view. Considering (1.1_ε) as a dynamical system on a compact manifold given by the torus $\mathbb{T}^3$, it is known[3] that there exists a nonempty, convex set $\{\mu\}^\varepsilon_I \subset \mathscr{B}(\mathbb{T}^3)$ [$\mathscr{B}(\mathbb{T}^3)$ is the set of probability Borel measures on $\mathbb{T}^3$] to which the flow induced by (1.1_ε) is invariant. That is,

Let $\Phi^t_\varepsilon(x, \theta, s)$ be the flow defined by (1.1_ε):

$$\Phi^t_\varepsilon(x, \theta, s) := (x_\varepsilon(t, s), \theta_\varepsilon(t, s), s + t)$$

where $\{x_\varepsilon(t, s), \theta_\varepsilon(t, s)\}$ is the solution of (1.1_ε) subject to the initial data $\{x_\varepsilon(0, s) = x, \theta_\varepsilon(0, s) = \theta\}$. Then $\mu \in \{\mu\}^\varepsilon_I$ if and only if $\forall \psi \in C^0(\mathbb{T}^3)$

$$\int_{\mathbb{T}^3} \psi\, d((\Phi^t_\varepsilon)^* \mu) \equiv \int_{\mathbb{T}^3} (\psi \circ \Phi^t_\varepsilon)\, d\mu = \int_{\mathbb{T}^3} \psi\, d\mu \qquad \forall t \in \mathbb{R} \tag{1.2}$$

i.e., $(\Phi^t_\varepsilon)^* \mu = \mu$ $\forall t \in \mathbb{R}$, where $(\Phi^t_\varepsilon)^*$ is defined by the identity sign in (1.2).

By the Krein–Milman theorem, the set of extreme points in $\{\mu\}^\varepsilon_I$ is nonempty. These are the ergodic measures, which we denote by $\{\mu\}^\varepsilon_e \subset \{\mu\}^\varepsilon_I$. For each $\mu \in \{\mu\}^\varepsilon_e$ we deduce by the Birkhoff ergodic theorem that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \delta(\Phi^t_\varepsilon(z))\, dt = \mu \tag{1.3}$$

holds a.s. $z \in \mathbb{T}^3$. Here δ is the Dirac δ function and the convergence is understood in the weak * sense (C^*-topology). The domain of attraction associated with $\mu \in \{\mu\}^\varepsilon_e$ is given by the set $z \in \mathbb{T}^3$ for which the limit (1.3) holds and is denoted by $\mathscr{D}(\mu)$. The measure μ is called a physical measure if $\text{meas}(\mathscr{D}(\mu)) > 0$, where $\text{meas}(\cdot)$ is the normalized Lebesgue measure on $\mathbb{T}^3$. Notice that, in general, μ is singular with respect to the Lebesgue measure, so we cannot conclude $\text{meas}(\mathscr{D}(\mu)) > 0$ from the Birkhoff ergodic theorem. In general, it is very difficult to prove that a singular ergodic measure of a smooth dynamical system is a physical one (unless it is supported on an attracting point or a limit cycle). Another definition of an invariant measure with a "physical significance" can be found in Eckmann.[(3),2]

For each $z \in \mathbb{T}^3$, the set $\{\mu\}_z^\varepsilon$ of asymptotic measures associated with z is defined by

$$\{\mu\}_z^\varepsilon = \left\{\mu \in \mathscr{B}(\mathbb{T}^3); \exists\{T_n\} \to \infty; \lim_{n\to\infty} \frac{1}{T_n}\int_0^{T_n} \delta(\Phi_\varepsilon^t(z))\, dt = \mu\right\}$$

Obviously, $\{\mu\}_z^\varepsilon \subset \{\mu\}_I^\varepsilon$. (But $\{\mu\}_z^\varepsilon$ are not necessarily ergodic measures for a given $z \in \mathbb{T}^3$.)

We now extend the definition $\mathscr{D}(\mu)$ to $\mathscr{D}(\{\mu\}_\imath^\varepsilon)$, where $\{\mu\}_\imath^\varepsilon$ is an arbitrary subset of the ergodic measures: $\{\mu\}_\imath^\varepsilon \subset \{\mu\}_e^\varepsilon$,

$$\mathscr{D}(\{\mu\}_\imath^\varepsilon) = \{z \in \mathbb{T}^3; \{\mu\}_z^\varepsilon \subseteq \text{Conv}(\{\mu\}_\imath^\varepsilon)\}$$

where $\text{Conv}(\cdot)$ means the convex hull. Notice that

$$\mathscr{D}(\{\mu\}_\imath^\varepsilon) \supseteq \bigcup_{\mu \in \{\mu\}_\imath^\varepsilon} \mathscr{D}(\mu)$$

where the inclusion above is generally a strict one.

Let $\mathscr{R} \subset \mathbb{S}$ be the set of resonant points associated with (1.1_ε).

Definition 1.1. Given $0 \leqslant \delta \ll 1$ and an open neighborhood $\mathscr{U}_\delta \supset \mathscr{R}$, $\text{meas}(\mathscr{U}_\delta) \leqslant \delta$, (1.1_ε) is said to admit a resonant trapping with respect to δ if there exists a set $\{\mu\}_\imath^\varepsilon \subset \{\mu\}_e^\varepsilon$ for which:

(a) $\mu(\mathscr{U}_\delta) \geqslant 1 - \delta \ \forall \mu \in \{\mu\}_\imath^\varepsilon$.

(b) $\text{meas}(\mathscr{D}(\{\mu\}_\imath^\varepsilon)) \geqslant 1 - \delta$.

Let (1.1_ε) admit a resonant trapping with respect to δ. Then a simple argument based on the weak * compactness of $\mathscr{B}(\mathbb{T}^3)$ yields the existence of a closed set $K \subset \mathscr{D}(\{\mu\}_\imath^\varepsilon)$, $\text{meas}(K) \geqslant 1 - 2\delta$, and a time $T(K) > 0$ for which

$$\frac{1}{T}\text{meas}\{t; 0 \leqslant t \leqslant T, \Phi_\varepsilon^t(z) \in \mathscr{U}_\delta\} \geqslant 1 - 2\delta$$

[2] A nontrivial example of a physical invariant measure for a smooth dynamical system possessing an axiom-A attractor is given by the SRB measure.[(2)]

provided $T > T(K)$ and $z \in K$. Thus, a resonant behavior will be observed for most of the time and for the majority of the initial data, provided δ is sufficiently small. Our goal is to find conditions on $\omega(\cdot)$ and $g(\cdot, \cdot, \cdot)$ so that for any $\delta > 0$, it is possible to find small enough $\varepsilon > 0$ for which (1.1_ε) admits a resonant trapping with respect to δ.

In Section 2 we pose some simplifying assumptions on (1.1_ε), reducing the system to a pair of ordinary differential equations on the 2D torus $\mathbb{T}^2 \equiv \mathbb{S} \otimes \mathbb{T}$. In Section 3 we classify the set of resonant points and introduce our main theorems. Section 4 reduces the problem to the question of the integrability of a logarithmic singularity with respect to a certain invariant measure ν, induced by the return map on a transversal loop in $\mathbb{T}^2$. After proving some technical results in Section 5, we complete the proof of our main theorems in Section 6. Some technical proofs are given in Appendixes A–C.

2. SIMPLIFICATION AND REDUCTION

Unfortunately, the problem as stated is beyond the capability of our analysis. This is partly due to the fact that the resonant set $\mathscr{R}$ is, in general, a dense set in $\mathbb{S}$. Thus, we impose a simplifying assumption:

$$g = \hat{g}(x, k\theta + lt)$$

$\hat{g}(\cdot, \cdot)$ is 1-periodic in both variables, and $\Omega(x) \equiv k\omega(x) + l$ admits a finite number of simple roots $x_1, ..., x_n$:

$$\Omega(x_j) = 0, \qquad \frac{d}{dx}\Omega(x_j) \neq 0 \qquad \text{for} \quad j = 1, ..., n$$

Therefore, the resonant set $\mathscr{R} = \{x_1, ..., x_n\} \subset \mathbb{S}$ is a finite set.

Introducing $y = k\theta + ls$, we find that (1.1_ε) is reduced in

$$\begin{aligned} \frac{dx}{dt} &= \varepsilon \hat{g}(x, y) \\ \frac{dy}{dt} &= \Omega(x) \end{aligned} \tag{2.1_ε}$$

Such a reduction enables us to restrict ourselves to the two-dimensional tours $\mathbb{T}^2$. The averaged equation corresponding to (2.1_ε) is given by

$$\frac{d}{dt}\bar{x} = \varepsilon \bar{g}(\bar{x}) \tag{2.2}$$

where

$$\bar{g}(x) \equiv \int_0^1 \hat{g}(x, y)\, dy$$

As before, we write $\mathbb{T}^2 = \mathbb{S} \otimes \mathbb{T}$, where $x \in \mathbb{S}$, $y \in \mathbb{T}$.

There are two generic possibilities for $\bar{g}$:

(A) $\bar{g}(x) \neq 0$ on $\mathbb{S}$.

(B) $\bar{g}(x) = 0$ admits a finite number of transversal roots on $\mathbb{S}$.

Assume first $\mathscr{R} = \varnothing$ [i.e., $\Omega(\cdot) = 0$ has no rots on $\mathbb{S}$]. Then (2.1_ε) represents a vector field on the torus with no singular points. In case (A) it is easy to see that $\{\mu\}_I^\varepsilon$ converges weakly, as $\varepsilon \to 0$, to a single measure given by the normalized density $C\bar{g}^{-1}(x)$ on $\mathbb{T}^2$. In Case (B), (2.1_ε) admits a finite set of limit cycles in a neighborhood of $\{\bar{x}_i\} \otimes \mathbb{T}$, where $\bar{x}_i$ are the roots of $\bar{g}(x) = 0$. These limit cycles contain the nonwandering set of (2.1_ε), and $\{\mu\}_I^\varepsilon$ consists of measures supported on those limit cycles.

Assume now

$$\mathscr{R} = (x_1, ..., x_n) \neq \varnothing$$

Then, considering (2.1_ε) with $\varepsilon = 0$, we obtain circles of fixed points at the resonant points $\{x_i\} \oplus \mathbb{T}$. For $\varepsilon \neq 0$ those circles break down into a finite set of saddle nodes in a $O(\varepsilon^{1/2})$ neighborhood of the resonances. We refer to the above neighborhood as the resonance zone, and to the invariant measures supported at those critical points as the resonant measures $\{\mu\}_r^\varepsilon$. The object of our analysis is to study $\mathscr{D}(\{\mu\}_r^\varepsilon)$ in accordance with the definition of resonant trapping.

We can immediately omit case (B) above from our analysis. Indeed:

Lemma 2.1. Assume $\mathscr{R} \neq \varnothing$ and $\bar{g}(\cdot) = 0$ admits a finite (nonzero) number of transverse roots $(\bar{x}_1, ..., \bar{x}_k)$. If $R \cap (\bar{x}_1, ..., \bar{x}_k) = \varnothing$, then[3]

$$\lim_{\varepsilon \to 0} \operatorname{meas}(\mathscr{D}(\{\mu\}_r^\varepsilon)) = 0$$

In particular, for ε small enough, (2.1_ε) does not admit δ resonant trapping for any fixed $\delta > 0$ (independent of ε).

The proof of Lemma 2.1 is given in Appendix 1.

From now on we assume

$$\bar{g}(\cdot) > 0 \quad \text{on} \quad \mathbb{S} \tag{2.3}$$

[3] Recall that $\{\mu\}_r^\varepsilon$ is, by Definition 1.1, the set of ergodic measures supported in a neighborhood of the resonant set $\mathscr{R}$.

3. CLASSIFICATION OF RESONANT POINTS

It is convenient to scale the time by $\varepsilon^{1/2}$, so that (2.1_ε) takes the form

$$\begin{aligned} \dot{x} &= \varepsilon^{1/2}\hat{g}(x, y) \\ \dot{y} &= \varepsilon^{-1/2}\Omega(x) \end{aligned} \tag{3.1$_\varepsilon$}$$

Let $x_j \in \mathscr{R}$ and let $G(x_j, y)$ be the primitive of $-\hat{g}(x_j, \cdot)$:

$$G(x_j, y) := -\int^y \hat{g}(x_j, \zeta)\, d\zeta$$

Set

$$\hat{x} = \frac{x - x_j}{\varepsilon^{1/2}}$$

Then

$$\begin{aligned} \dot{\hat{x}} &= -\frac{\partial}{\partial y} G(x_j, y) + \varepsilon^{1/2} \frac{\partial}{\partial x} \hat{g}(x_j, y)\hat{x} + O(\varepsilon) \\ \dot{y} &= \Omega'(x_j)\hat{x} + \frac{1}{2}\varepsilon^{1/2}\Omega''(x_j)\hat{x}^2 + O(\varepsilon) \end{aligned} \tag{3.2}$$

Substituting $\varepsilon = 0$ in (3.2), we obtain a second-order equation:

$$\ddot{y} + \Omega'(x_j)\frac{\partial}{\partial y} G(x_j, y) = 0 \tag{3.3}$$

Assuming without loss of generality that $\Omega'(x_j) > 0$, then the minimal points of $G(x_j, \cdot)$ correspond to nodes, while its maximal points corresponds to saddles of (3.3). For simplicity we assume that $G(x_j, \cdot)$ admits a unique pair of maximum–minimum points on the unit interval. Let y_0 be the maximal point of $G(x_j, \cdot)$. Figure 1 demonstrates the homoclinic phase orbit of (3.3).

Let

$$\mathscr{H}(\hat{x}, y) = \tfrac{1}{2}\Omega'(x_j)\hat{x}^2 + G(x_j, y)$$

Then, via (3.2),

$$\frac{d\mathscr{H}}{dt} = \varepsilon^{1/2}[\Omega'(x_j)\, \hat{g}_x(x_j, y)\hat{x}^2 - \hat{g}(x_j, y)\, \Omega'(x_j)\hat{x}^2] + O(\varepsilon) \tag{3.4}$$

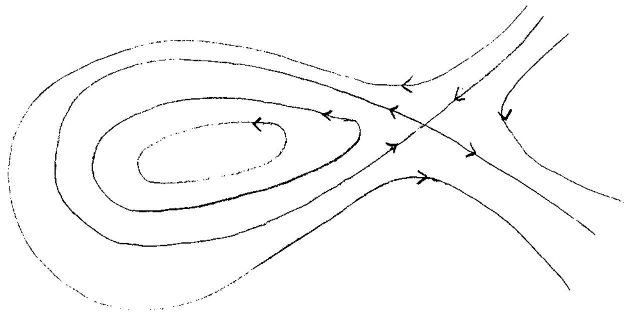

Fig. 1. Flow curves of Eq. (2.1_ε), including resonance, approximated to order $\varepsilon^{1/2}$ [consistent with Eq. (3.3)].

where $\hat{g}_x$ stands for $(\partial/\partial x)\,\hat{g}$. The increment of $\mathscr{H}$ along the homoclinic orbit of (3.3) is given by integrating the right-hand side of (3.4) along the orbit. To leading order $[O(\varepsilon^{1/2})]$, it is given by $\varepsilon^{1/2}M$, where $M := M(x_j)$ is the Melnikov function associated with the resonance at x_j:

$$M(x_j) \equiv \frac{1}{\Omega'(x_j)} \oint \hat{g}_x(x_j, y)[E - G(x_j, y)]^{1/2}\, dy \tag{3.5}$$

where $E := G(x_j, y_0)$ and the integral is carried along the homoclinic orbit

$$\{(\hat{x}, y);\, \mathscr{H}(\hat{x}, y) = G(x_j, y_0)\}$$

represented by the phase-space variables $\hat{x}$, y. (cf. ref. 8, Chapter 1.3).

Definition 3.1. A resonant point $x_j \in \mathbb{S}$ is said to be an attracting (repelling) resonance if $M(x_j) < 0$ $[M(x_j) > 0]$.

The phase flow of a repelling resonance is depicted in Fig. 2.

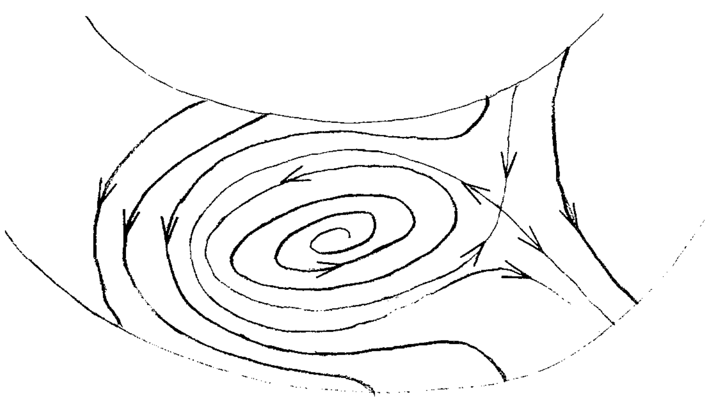

Fig. 2. Structure of a repelling resonance.

In particular, a resonance is repelling if $\hat{g}_x(x_j, \cdot) > 0$ on $\mathbb{T}$.

Definition 3.2. Let $\{x_j\} \in \mathscr{R}$ and let $\{y_0\}$ be the corresponding saddle point (i.e., a local maximum of $G(x_0, \cdot)$). Then the *trace* of the resonance $\{x_j\}$ is defined as the sign of $\hat{g}_x(x_j, y_0)$.

Remark. The contribution of $\hat{g}_x(x_j, y)$ near the saddle point $y = y_0$ is negligible to the integral (3.5), since $[E - G(x_j, y_0)]^{1/2} = 0$ by definition. Thus, a resonance point $\{x_j\}$ can be both repelling and of negative trace.

Let $\bar{\alpha}(\varepsilon)$ be the rotation number associated with (2.1_ε) (see Section 4).

Lemma 3.1. Assume all points in $\mathscr{R}$ are of the same type (either all attracting or else all repelling). Assume further that

$$\Lambda \equiv \int_0^T \Omega(\bar{x}(s))\, ds \neq 0 \tag{3.6}$$

where $\bar{x}(\cdot)$ is a solution of the averaged equation (2.2) and T is its period on $\mathbb{S}$ [i.e., $\bar{x}(T) = \bar{x}(0) + 1$]. Then for any $r \in [0, 1)$ there exists a sequence $\varepsilon_n \downarrow 0$ such that

$$\bar{\alpha}(\varepsilon_n) = r, \qquad n = 1, 2, \ldots \tag{3.7}$$

A discussion on the rotation number $\alpha(\varepsilon)$ associated with (2.1_ε) and the proof of Lemma 3.1 are given in Section 4.

In Section 5 we proceed and prove some technical results, while in Section 6 we present our main results:

Theorem 3.1. Assume:

(a) All resonances are repelling.

(b) All corresponding traces are nonpositive and at most one of them is strictly negative.

Assume further that ε is small enough and $\bar{\alpha}(\varepsilon)$ is a number of Liouville type (i.e., represented by a continued-fraction expansion where all its digits are uniformly bounded).

Then all ergodic measures of (2.1_ε) are supported on the critical points of the vector field associated with (2.1_ε).

Evidently, Theorem 3.1 implies that (2.1_ε) admits a resonant trapping for any $\delta > C\varepsilon^{1/2}$, where $C > 0$ is a constant independent of ε, provided ε satisfies (3.7). Indeed, the set of critical points of (2.1_ε) is contained in a $\varepsilon^{1/2}$ neighborhood of the resonance, so Theorem 3.1 guarantees that $\{\mu\}_I^\varepsilon$ is supported in a δ neighborhood of the resonant zone. Thus, we may set $\{\mu\}_i^\varepsilon = \{\mu\}_I^\varepsilon$ in Definition 1.1. Then, Lemma 3.1 guarantees that resonance trapping holds for infinite (in fact, uncountable) value of $\varepsilon > 0$.

To show that the assumptions of Theorem 3.1 are not completely redundant, we introduce:

Theorem 3.2. Let assumption (a) in Theorem 3.1 hold, and replace (b) by:

(b′) All traces are strictly positive.

Then, for any ε for which $\bar{\alpha}(\varepsilon)$ is irrational, there exists an ergodic measure $\tilde{\mu} \in \{\mu\}_e^\varepsilon$ *not* supported on any of the critical points of (2.1_ε) and $\mathscr{D}(\tilde{\mu})$ contains all points of $\mathbb{T}^2$, excluding the critical points and their stable manifolds.

Since the set of critical points and their stable manifolds is of small Lebesgue measure $[O(\varepsilon^{1/2})$ at most, or 0 if no stable critical point exists on $\mathbb{T}^2]$, it follows, under the assumptions of Theorm 3.2, that (2.1_ε) does *not* admit a resonant trapping for any $\delta > 0$ and ε satisfying (3.7).

4. THE CIRCLE MAP

In order to define the circle map associated with (2.1_ε), we introduce the following result.

Lemma 4.1. Given $x_0 \notin \mathscr{R}$ and ε small enough, there exists a closed loop C^ε in an ε neighborhood of $\{x_0\} \oplus \mathbb{T}$, homologous to $\mathbb{T}$, such that the flow of (2.1_ε) traverses C^ε.

The proof follows by an application of a Bogoliubov near-identity transformation on (2.1_ε) in a neighborhood of $\{x_0\} \otimes \mathbb{T}$ and by our standing assumption (2.3). For ε small enough we may assume, without limiting the generality,

$$C^\varepsilon := \{x_0, \mathbb{T}\}$$

where $x_0 \in \mathbb{S}$ is any point outside the resonance zone for which $\bar{g}(x_0) \neq 0$.

We now view (2.1_ε) as an equation on the covering space $S \otimes \mathbb{R}^1$ (i.e., $x \equiv x \bmod 1$, $-\infty < y < \infty$). For each $y \in \mathbb{R}^1$ we consider the orbit starting at $\{x_0, y\} \in C^\varepsilon$. If the orbit intersects C^ε at a positive time, we define $\tilde{F}_\varepsilon(y)$ as the value of the y coordinate of the orbit at the intersection point. The index ε will occasionally be omitted.

Lemma 4.2. Assume:

(A) All resonant points of (2.1_ε) are repelling.

Then

$$\tilde{F}\colon \quad \mathbb{R}^1 \to \mathbb{R}^1 \tag{4.1}$$

is defined at any $y \in \mathbb{R}^1$, excluding a set $\{y_1 \cdots y_n\} + \mathbb{Z}$, where $\{y_1 \cdots y_n\} \in [0, 1)$. The map $\tilde{F}$ is strictly monotone and satisfies

$$\tilde{F}(y+1) = \tilde{F}(y) + 1 \tag{4.2}$$

Moreover, $d\tilde{F}/dy$ is continuous and positive on $\mathbb{R}^1 \backslash \{\{y_1 \cdots y_n\} + \mathbb{Z}\}$.

(B) All resonant points of (2.1_ε) are attracting.

Then $\tilde{F}$ is defined at any $y \in \mathbb{R}^1$, excluding a set $L_1 \cup L_2 \cup \cdots \cup L_n + \mathbb{Z}$, where $\{L_j\}_{j=1}^n \subset [0, 1)$ are disjoint closed subintervals. The function $\tilde{F}$ satisfies (4.2) and $d\tilde{F}/dy$ is continuous and positive on $\mathbb{R}^1 \backslash \{\bigcup_1^n L_j + \mathbb{Z}\}$. Moreover,

$$\lim_{y \searrow L_j^+} \frac{d\tilde{F}}{dy} = \lim_{y \nearrow L_j^+} \frac{d\tilde{F}}{dy}$$

where L_j^+ (L_j^-) is the right (left) endpoint of L_j.

Proof of Lemma 4.2. In case (A), the set $\{y_1 \cdots y_n\} \subset \mathbb{T}$ is made up of those points for which $(x_0. y_i) \in C^\varepsilon$ belongs to the first intersection of C^ε with the stable manifold of one of the saddle points of (2.1_ε). In case (B), the intervals L_j are given by the first intersection of C^ε with the stable manifold of the *stable nodes* of (2.1_ε) (Fig. 3).

The existence of $\tilde{F}$ and the relation (4.2) is obvious, in both cases, by the existence and uniqueness theorems for ordinary differential equations.

Let $x = P(x, y, t)$, $y = Q(x, y, t)$ be the solution of (2.1_ε) subject to

$$P(x, y, 0) = x, \qquad Q(x, y, 0) = y$$

If $\tilde{F}(y)$ is defined, let $t(y) < \infty$ be the first time at which the orbit intersects C^ε. Then, by definition,

$$P(x_0, y, t(y)) = x_0, \qquad Q(x_0, y, t(y)) = \tilde{F}(y) \tag{4.3}$$

Since

$$\frac{\partial P}{\partial t}(x_0, y, t(y)) = \hat{g}(x_0, F(y)) \tag{4.4}$$

and $\hat{g}(x_0, \cdot) > 0$ by assumption, the smoothness of $t(\cdot)$ follows via the first equality of (4.3) and the implicit function theorem. Differentiating both equalities of (4.3) with respect to y, using (4.4) and its corresponding equation for $\partial Q/\partial t$, we obtain, after some manipulation,

$$\frac{\partial \tilde{F}}{dy} = \hat{g}(x_0, \tilde{F}(y))^{-1} [g(x_0, \tilde{F}(y))\, Q_y(x_0, y, t(y)) - \varepsilon^{-1} \Omega(x_0)\, P_y(x_0, y, t(y))] \tag{4.5}$$

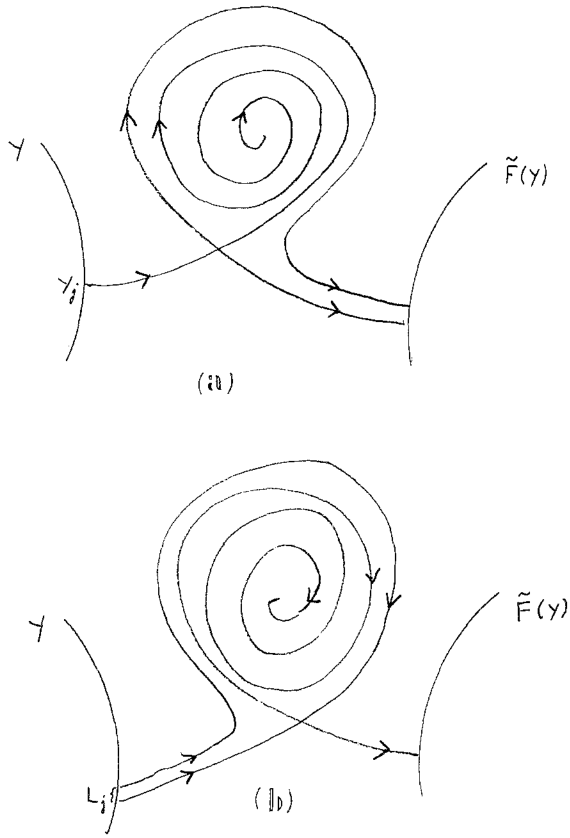

Fig. 3. Local structure of the lift $\tilde{F}$ corresponding to (a) repelling resonance; (b) attracting resonance.

Define

$$W(t) = -\hat{g}(P(x_0, y, t), Q(x_0, y, t))\, Q_y(x_0, y, t) + \frac{1}{\varepsilon}\, \Omega(P(x_0, y, t))\, P_y(x_0, y, t) \tag{4.6}$$

and observe that $W(0) = -\hat{g}(x_0, y) < 0$, while

$$W(t(y)) = -[\hat{g}(x_0, \tilde{F}(y))\, Q_y(x_0, y, t(y)) - \varepsilon^{-1}\Omega(x_0)\, P_y(x_0, y, t(y))]$$

Notice that $W(\cdot)$ is the Wronskian of the variational equation (2.1_ε), centered at the orbit $(P(\cdot), Q(\cdot))$. Hence $\text{sign}[W(t(y))] = \text{sign}[W(0)] = -1$ and the positiveness of $d\tilde{F}/dy$ follows via (4.5). ∎

Definition. A monotone nondecreasing function $\tilde{F}: \mathbb{R}^1 \to \mathbb{R}^1$ which satisfies (4.2) is called a lift of a circle map.

We now extend the definition of $\tilde{F}$ to $\mathbb{R}^1$.

(A) Assume all resonances are repelling. Let I_j be the open interval

$$I_j = (\lim_{y \nearrow y_j} \tilde{F}(y), \lim_{y \searrow y_j} \tilde{F}(y)) \tag{4.7}$$

where $y_j \in [0, 1)$ is a point of discontinuity of $\tilde{F}$. We define $\tilde{F}$ on the whole real line by assigning an arbitrary value to $\tilde{F}$ at each y_j. Since we want to preserve the monotonicity of $\tilde{F}$, we choose $\tilde{F}(y_j) \in I_j$ for each discontinuity point $y_j \in [0, 1)$, and extend the definition to $\mathbb{R}^1$ is accordance with (4.2). Thus, $\tilde{F}$ is extended to a lift of a circle map and is discontinuous and strictly monotone.

(B) Assume all resonances are attracting. On each interval L_j we assign the value $\tilde{F}(y)$, which agrees with $\tilde{F}$ on the endpoints $L_j^{\pm}$:

$$\tilde{F}(y) := \tilde{F}(L_j^+) = \tilde{F}(L_j^-); \qquad y \in L_j$$

(see Fig. 3b). Thus, $\tilde{F}$ is extended to a lift of a circle map and is continuous and weakly monotone.

We define the rotation number $\alpha(\tilde{F}) \in \mathbb{R}$:

$$\alpha(\tilde{F}) = \lim_{k \to \infty} \frac{\tilde{F}^k(y)}{k} \tag{4.8}$$

Here $\tilde{F}^k$ stands for the kth iterate of $\tilde{F}$.

For $\tilde{F}$ given by a lifting of a circle homeomorphism, the rotation number is known to exist and is independent of the choice of $y \in \mathbb{R}^1$ in (4.8) (ref. 1, Theorem on p. 102). Moreover, the rotation number is rational iff the associated homeomorphism admits a periodic orbit. In our case, $\tilde{F}$ is *not* a lift of a circle homeomorphism since it either admits discontinuity points at $\{y_1, \ldots, y_n\} + \mathbb{Z}$ in the repelling case (A) or is constant on the intervals $L_j + \mathbb{Z}$ in the attracting case (B). The following proposition is a straightforward generalization of the rotation number theorem to any lift $\tilde{F}$:

Proposition 4.1. The limit (4.8) exist for any lift of a circle map. If $\tilde{F}$ admits a point of discontinuity, then $\alpha(\tilde{F})$ is independent of the values assigned to $\tilde{F}$ at the points of discontinuity, provided the monotonicity of $\tilde{F}$ is preserved. The limit (4.8) is rational iff the induced circle map admits a periodic orbit for some choice of $\tilde{F}(\cdot)$ at points of discontinuity.

Definition 4.1. The rotation number associated with (2.1_ε) is denoted by

$$\alpha(\varepsilon) := \alpha(\tilde{F}_\varepsilon)$$

By Proposition 4.1, $\alpha(\varepsilon)$ is well defined for both the attracting (B) and repelling (A) cases.

Our next step is to study the continuity of $\alpha(\cdot)$ as a function of ε. For continuous lifts, the rotation number is continuous with respect to the C^0 topology (ref. 1, Chapter 3.12). Thus, the continuity of $\alpha(\cdot)$ is obvious in the case of attreacting resonance (B).

In the repelling case, however, $\tilde{F}$ admits discontinuity points, so the C^0 topology is not appropriate. Thus, we have to consider an alternative topology. For this purpose we consider the mapping $\mathscr{F}$ from the set of monotone-increasing lifts to the set of monotone-nondecreasing continuous lifts, given by

$$\mathscr{F}(\tilde{F})_{(y)} = \tilde{F}^{-1}(y) \qquad \text{if } \ y \in \text{Range } F$$

$$\mathscr{F}(\tilde{F})_{(y)} = y_j + k \qquad \text{if } \ y \in I_j + k, \quad \forall k \text{ integer}$$

where I_j is given in (4.7).

Notice that $\mathscr{F}$ transforms $\tilde{F}$ associated with (2.1_ε) into $\tilde{F}$ associated with *the time reversal* of (2.1ε). Note that time reversal transforms a repelling resonance into an attractive one and $\alpha(\tilde{F}) = -\alpha(\mathscr{F}(\tilde{F}))$. Combining these facts with the remark below Definition 4.1, we obtain the following result.

Lemma 4.3. The rotation number restricted to the set of strictly monotone lifts is a continuous function with respect to the topology induced by pulling back the C^0 topology via $\mathscr{F}(\cdot)$.

Remark. The definition of the above topology cannot be extended to circle maps which are both discontinuous and admit constant values over subintervals. Indeed, for such maps the rotation number is not necessarily continuous in any reasonable topology. As an example, let $0 < \tilde{y}_1 < \tilde{y}_2 < 1$ and consider the map

$$\tilde{F}_\beta(y) = \begin{cases} \beta + y & \text{for } \ y \in [0, 1] \backslash (\tilde{y}_1, \tilde{y}_2] \\ \beta + \tilde{y}_1 & \text{for } \ y \in (\tilde{y}_1, \tilde{y}_2) \\ \tilde{z} & \text{for } \ y = \tilde{y}_2 \end{cases}$$

where $\tilde{z} \in [\beta + \tilde{y}_1, \beta + \tilde{y}_2]$. Complete $\tilde{F}_\beta$ over the reals via (4.2) (cf., Fig. 4).

It is easy to see that the rotation number $\alpha(F_\beta)$ admits only rational values on the one hand, and is not a constant with respect to β on the

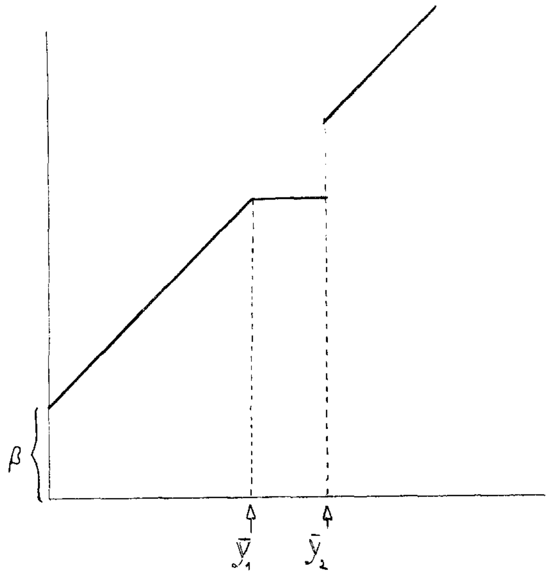

Fig. 4. Graph of $F_\beta(\cdot)$, restricted to $0 \leqslant y \leqslant 1$.

other hand. Thus, $\alpha(F_\beta)$ is not continuous with respect to the parameter β. Thus, Lemma 4.3 cannot be extended to the mixed case where both attracting and repelling resonances exist simultaneously.

Corollary 4.1. If all resonant points of (2.1_ε) are of the same type (either all repelling or all attracting), then $\alpha(\cdot)$ is a continuous function of ε.

Next, we consider the behavior of $\alpha(\varepsilon)$ as $\varepsilon \to 0$:

Lemma 4.4. Let $\bar{x}(t)$ be the solution of the averaged equation (2.2) with period T, i.e.,

$$\dot{\bar{x}} = \bar{g}(\bar{x}) \tag{2.2'}$$

$$\bar{x}(t+T) = \bar{x}(t) + 1 \qquad \forall t \in \mathbb{R}$$

Assume

$$\int_0^T \Omega(\bar{x}(s))\, ds > 0 \quad (<0) \tag{4.9}$$

Then

$$\lim_{\varepsilon \to 0} \alpha(\varepsilon) = \infty \quad (-\infty) \tag{4.10}$$

Definition 4.2. The rotation number associated with the *circle map* induced by $\tilde{F}_\varepsilon$ is

$$\bar{\alpha}(\varepsilon) := \alpha(\varepsilon) \bmod 1 \tag{4.11}$$

With the above definition at hand, we are in a position to prove Lemma 3.1 (Section 3):

Proof of Lemma 3.1. By Corollary 4.1, $\alpha(\varepsilon)$ is continuous in both the repelling and attracting cases. Then (4.10) and (4.11) yield (3.7). ∎

Proof of Lemma 4.4. It is enough to show that for some $y_0 \in [0, 1)$

$$\lim_{\varepsilon \to 0} \tilde{F}_\varepsilon(y_0) = \infty \quad (-\infty) \tag{4.12}$$

We use a simplified version of Neistadt's theorem[9]: Given $\delta_1 > 0$, $\bar{T} > 0$, we can find $\bar{\varepsilon} > 0$ so that for some $(x_0, y) \in C^\varepsilon$,

$$\sup_{0 \leqslant t \leqslant \bar{T}/\varepsilon} |x(t) - \bar{x}(t)| < \delta_1 \tag{4.13}$$

provided $\varepsilon < \bar{\varepsilon}$. Here $\bar{x}(\cdot)$ is the solution of the averaged equation (2.2), $\bar{x}(0) = x_0$, and $x(\cdot)$ is the solution of (2.1_ε) subject to $x(0) = x_0$, $y(0) = y$.

Let $\bar{T} > 2T$. Then

$$y(t) = y_0 + \int_0^t \frac{\Omega(x(s))}{\varepsilon} ds = y_0 + \int_0^t \frac{\Omega(\bar{x}(s))}{\varepsilon} + O\left(\frac{\delta_1}{\varepsilon}\right)$$

provided $t \leqslant 2T/\varepsilon$.

Moreover, if $t(y_0)$ is the first intersection time of the orbit with C^ε for the above chosen y_0, then, by (4.12), $|T - t(y_0)| = O(\delta_1)$. Setting $\tilde{F}(y_0) = y(t(y_0))$, we have

$$\left| (\tilde{F}(y_0) - y_0) - \frac{1}{\varepsilon} \int_0^T \Omega(\bar{x}(s))\, ds \right| = O\left(\frac{\delta_1}{\varepsilon}\right)$$

Thus we can choose δ_1 so small that

$$\tilde{F}(y_0) - y_0 > \frac{1}{2\varepsilon} \int_0^T \Omega(\bar{x}(s))\, ds \quad \left[< \frac{1}{2\varepsilon} \int_0^T \Omega(\bar{x}(s))\, ds \right]$$

and (4.12) follows. ∎

We conclude this section with a technical lemma which describes the behavior of $\tilde{F}$ near a discontinues point.

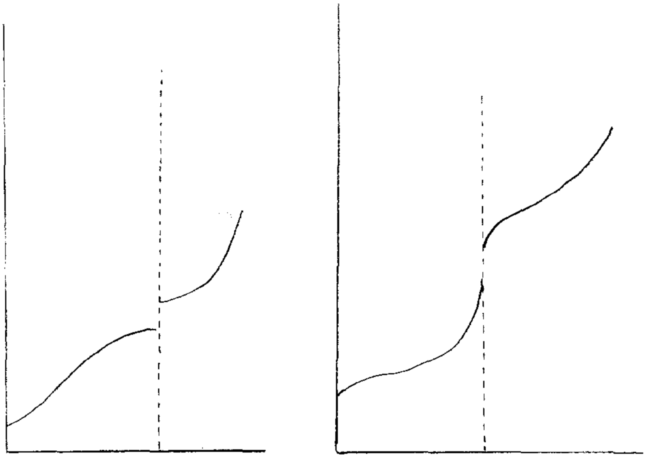

Fig. 5. Discontinuity points of $\tilde{F}$ corresponding to repelling resonances: (a) positive trace; (b) negative trace.

Lemma 4.5. Assume $y_0 \in C^\varepsilon$ is a discontinuity point of $\tilde{F}$, given by the first intersection of the stable manifold of a saddle point $(\hat{x}, \hat{y}) \in \mathbb{T}^2$ with C^ε. Assume $\lambda_2 > 0 > \lambda_1$ are the eigenvalues of the Jacobian matrix of (2.1_ε) at the saddle. Then

$$\begin{aligned} &\lim_{y \searrow y_0} |y - \tilde{F}(y_0 + 0)|^{-(\lambda_1 + \lambda_2)/\lambda_1} \frac{d}{dy} \tilde{F}(y) > 0 \\ &\lim_{y \nearrow y_0} |y - \tilde{F}(y_0 - 0)|^{-(\lambda_1 + \lambda_2)/\lambda_1} \frac{d}{dy} \tilde{F}(y) > 0 \end{aligned} \tag{4.14}$$

The proof of Lemma 4.5 is analogous to the proof given by Wiggins (ref. 10, Chapter 3.2). Figure 5 demonstrates the discontinuity of $\tilde{F}$ in the cases of positive and negative trace.

5. ERGODIC MEASURES

We assume throughout this section that all resonant points of (2.1_ε) are repelling. Let $\bar{\alpha}$ (cf. Definition 4.2) be the rotation number of the circle map F: $\mathbb{T} \to \mathbb{T}$:

$$F(y \bmod 1) := \tilde{F}(y) \bmod 1$$

where $F(y_i) \in I_i$ (mod 1) at the discontinuity points [see (4.7)]. Since F is an order-preserving map on the circle, we obtain Lemma 5.1 below as a direct generalization of Denjoy's theory of continuous, invertible circle maps (ref. 1, Chapter 3).

Lemma 5.1. If $\bar{\alpha}$ is irrational, then F is semiconjugate to the rigid rotation, $R_\alpha(y) = \alpha + y \bmod 1$, i.e., $\exists h$: $\mathbb{T} \to \mathbb{T}$, h is continuous (but not necessarily invertible) so that

$$R_\alpha \circ h = h \circ F$$

Moreover, the Borel measure

$$d\nu = dh$$

is the only probability measure on $\mathbb{T}$ which is F invariant.

Definition 5.1. Let

$$t: \quad \mathbb{S} \to \mathbb{S}$$

be the return time associated with (2.1_ε) on the transversal loop C^ε. Namely,

$$t(y) := \inf_{\delta > 0} \inf_{s > \delta} \{s;\, (x(s), y(s)) \in C^\varepsilon\}$$

where $(x(\cdot), y(\cdot))$ is the solution of (2.1_ε) starting at the point $\{y(0) = y, x(0) = x_0\} \in C^\varepsilon$.

Remark. It is easy to see that $t(\cdot)$ admits a logarithmic singularity at the discontinuity points of F.

Assume $t(\cdot)$ is ν-integrable and set

$$\bar{T} \equiv \oint_{C^\varepsilon} t(y)\, \nu(dy) < \infty \tag{5.1}$$

More generally, let $\phi \in C^0(\mathbb{T}^2)$ and define $P(\phi, \cdot)$ as a function on C^ε:

$$P(\phi, y) = \int_0^{t(y)} \phi(z_{(s)}^{(y)})\, ds \tag{5.2}$$

where $z_{(\cdot)}^{(y)}$ is the orbit of (2.1_ε) starting from $y \in C^\varepsilon$. Obviously, $P(\phi, y)$ is continuous on $C^\varepsilon \setminus \{y_1, \ldots, y_n\}$ and admits at most a logarithmic singularity at $\{y_1, \ldots, y_n\}$. Evidently

$$C^{-1} t(\cdot) < P(\phi, \cdot) < C t(\cdot)$$

for some $C < \infty$; hence, by (5.1)

$$-\infty < \oint_{C^\varepsilon} P(\phi, y)\, v(dy) < \infty \tag{5.3}$$

Define a Borel measure $\tilde{\mu} \in \mathscr{B}(\mathbb{T}^2)$ by

$$\int_{\mathbb{T}^2} \phi\, d\tilde{\mu} = \frac{1}{\bar{T}} \oint_{C^\varepsilon} P(\phi, y)\, v(dy); \qquad \forall \phi \in C^0(\mathbb{T}^2) \tag{5.4}$$

The following lemma is the cornerstone of all our main results.

Lemma 5.2. Assume $\bar{\alpha}(F)$ is irrational.

(a) If $t(\cdot)$ is v-integrable, then $\tilde{\mu}$ is an ergodic invariant measure of (2.1_ε) and is singular with respect to the invariant measures supported on critical points of (2.1_ε). Moreover, the domain of attraction $\mathscr{D}(\tilde{\mu})$ contains all the points in $\mathbb{T}^2$ excluding the critical points of (2.1_ε) and their stable manifolds.

(b) If $t(\cdot)$ is not v-integrable, then all ergodic measures of (2.1_ε) are supported on its critical points.

The proof of Lemma 5.2 is rather technical and we present it in Appendix B.

Corollary 5.1. If $\bar{\alpha}(\varepsilon)$ is irrational and the saddle points are all of positive trace (negative trace), then all ergodic invariant measures are supported on the saddle points iff

$$\oint \log \left[\frac{d}{dy} F(y) \right] dv(y) = \infty \quad (-\infty) \tag{5.5}$$

where the integral in (5.5) is over $C^\varepsilon \backslash \{ y_1 \cdots y_n \}$.

Proof. The singularity of $|\log[(d/dy)\, F(y)]|$ near the discontinuity points of F is of the same type as that of the return time $t(\cdot)$ (cf. Definition 5.1). Indeed, $|\log[(d/dy)\, F(y)]|$ admits a logarithmic singularity by Lemma 4.5. Thus, the corollary follows directly from Lemma 5.2. ∎

Lemma 5.3. Suppose $\alpha(\varepsilon)$ is irrational and the flow (2.1_ε) admits an ergodic measure $\tilde{\mu}$ not supported on any of its critical points.

Let $\hat{\lambda}_1, \hat{\lambda}_2$ be the Liapunov (characteristic) exponents associated with $\tilde{\mu}$. Then

$$\hat{\lambda}_1 = \hat{\lambda}_2 = \int_{C^\varepsilon} \log \left[\frac{d}{dy} F(y) \right] dv(y) = 0 \tag{5.6}$$

Remark. From Corollary 5.1 it follows that $|\log[(d/dy)\, F(y)]|$ is ν-integrable.

Proof of Lemma 5.3. We rewrite (2.1_ε) as

$$\frac{dq^t}{dt} = Q_\varepsilon(q^t)$$

where $q^t\colon \mathbb{T}^2 \to \mathbb{T}^2$ is the flow map associated with (2.1_ε). Let $J(q_t)$ be the Jacobian derivative of q^t. Then $\forall t > 0$

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \int_{\mathbb{T}^2} \operatorname{tr}(\nabla Q_\varepsilon)\, d\tilde{\mu} \tag{5.7}$$

Assume, without loss of generality, that the flow q^t is transverse to C^ε. By the assumption of the lemma and Corollary 5.1,

$$\bar{T} \equiv \int_{C^\varepsilon} t(y)\, d\nu(y) < \infty$$

Since $\operatorname{tr}(\nabla Q_\varepsilon)$ is a smooth function, we obtain from (5.2) and (5.4)

$$\int_{\mathbb{T}^2} \operatorname{tr}(\nabla Q_\varepsilon)\, d\tilde{\mu} = \frac{1}{\bar{T}} \int_{C^\varepsilon} \left[\int_0^{t(y)} \operatorname{tr}(\nabla Q_\varepsilon(q_s^{(y)}))\, ds \right] d\nu(y)$$

where $q_{(s)}^{(y)}$ is the orbit of (2.1_ε) starting at $(x_0, y) \in C^\varepsilon$. On the other hand,

$$\int_0^{t(y)} \operatorname{tr}(\nabla Q_\varepsilon(q_{(s)}^{(y)}))\, ds = \log J(q^{t(y)}) \tag{5.8}$$

From (4.5)

$$J(q^{t(y)}) = \frac{g(x_0, F(y))}{g(x_0, y)} \frac{dF}{dy} \tag{5.9}$$

Since ν is F invariant, (5.7)–(5.9) yield

$$\hat{\lambda}_1 + \hat{\lambda}_2 = \frac{1}{\bar{T}} \int_{C^\varepsilon} \log J(q^{t(y)})\, d\nu(y) = \frac{1}{\bar{T}} \int_{C^\varepsilon} \log \frac{d}{dy} F(y)\, d\nu(y) \tag{5.10}$$

To complete the proof of the lemma, we introduce the following result.

Theorem 5.1. Consider a continuous-time dynamical system and let μ be an ergodic invariant measure. If all characteristic exponents associated with μ are different from zero, then $\mu = \delta_p$, where p is a fixed point. If one of the characteristic exponents is zero and the rest are all positive or all negative, then either $\mu = \delta_p$ or μ is supported on a limit cycle.

For the proof of the theorem, see Eckmann (ref. 3, Section $\mathrm{III}_{\mathrm{D}}$) and references cited there.

By the assumption of the lemma, μ is not supported on a fixed point and since $\bar{\alpha}(\varepsilon)$ is irrational, there exists no limit cycle of (2.1_ε). Hence

$$\hat{\lambda}_1 = \hat{\lambda}_2 = 0 \tag{5.11}$$

Equation (5.6) now follows from (5.10) and (5.11). ∎

6. PROOF OF THE MAIN THEOREMS

In this section we will first prove Theorem 3.2 and then Theorem 3.1. Theorem 3.2 follows from the next result.

Lemma 6.1. Under the conditions of Theorem 3.2

$$\int_{C^\varepsilon} \log \frac{dF}{dy} \, dv \leqslant 0 \tag{6.1}$$

Indeed, by Lemmas 4.2 and 4.5, $\log(dF/dy)$ is bounded from below on $\mathbb{T}$. Hence (6.1) yields the v-integrability of $|\log(dF/dy)|$. Then, using Lemma 4.5 again, we obtain the integrability of the logarithmic singularity at any of the discontinuous points of F. This yields the integrability of $t(\cdot)$ and the result follows by Lemma 5.2. Notice that, usinig Lema 5.3, we obtain from (6.1)

$$\int_{C^\varepsilon} \log \frac{dF}{dy} \, dv = 0$$

In a similar way, Theorem 3.1 is a consequence of the following result.

Lemma 6.2. Under the conditions of Theorem 3.1

$$\int_{C^\varepsilon} \log \frac{dF}{dy} \, dv < 0 \tag{6.2}$$

Indeed, if there exists an invariant ergodic measure μ not supported on the singular points of (2.1_ε), then we obtain a contradiction to Lemma 5.3 and Theorem 5.1.

Proof of Lemma 6.1. Assume

$$\oint_{C^\varepsilon} \log \frac{d}{dy} F(y) \, dv \geqslant \beta > 0 \tag{6.3}$$

By monotone convergence, there exists $\theta > 0$ large enough for which

$$\oint_{C^\varepsilon} \log\left[\theta \Lambda\left(\frac{d}{dy} F\right)\right] dv > \frac{2\beta}{3}$$

Since F is uniquely ergodic and $\theta\Lambda[(d/dy)F] \in C^0(\mathbb{T})$, then for each $n \geqslant N$ large enough and each $y \in \mathbb{T}$

$$\sum_{j=0}^{n-1} \log\left[\theta\Lambda \frac{d}{dy} F(F^j(y)\right] > \frac{\beta}{2} n$$

But

$$\begin{aligned}
\log \frac{d}{dy} F^n(y) &= \sum_{j=0}^{n-1} \log \frac{d}{dy} F(F^j(y)) \\
&\geqslant \sum_{j=0}^{n-1} \log\left[\theta\Lambda \frac{d}{dy} F(F^j(y))\right] \geqslant \frac{\beta n}{2} \\
&\Rightarrow \frac{d}{dy} F^n(y) \geqslant e^{\beta n/2} \qquad \forall y \in \mathbb{T}
\end{aligned} \tag{6.4}$$

However, since F^n maps the unit circle into itself

$$\int_{C^\varepsilon} \frac{d}{dy} F^n(y)\, dy < 1 \qquad \forall n \quad \exists \mathbb{N}$$

which contradicts (6.4); hence $\beta = 0$. ∎

For the proof of Lemma 6.2, we need the following result.

Lemma 6.3. Let $\{y_1 \cdots y_n\}$ be the discontinuity points of F. Let $I_j^{(0)} \equiv I_j$ as defined in (4.7). Set

$$I_j^{(k)} = F^k(I_j^{(0)})$$

If the rotation number $\bar{\alpha}(F)$ is irrational, then:

(a) $y_j \notin I_j^{(k)}$ $\forall k \geqslant 0$.

(b) $I_j^{(k)} \cap I_j^{(1)} = \varnothing$ if $k \neq 1$, $1 \leqslant j \leqslant n$.

(c) $\exists m$, $1 \leqslant m \leqslant n$, such that

$$I_m^{(k)} \cap \{y_1 \cdots y_n\} = \varnothing \qquad \forall k \geqslant 0$$

and $I_m^{(k)}$, $k \geqslant 0$, are connected intervals.

Proof of Lemma 6.3. Suppose $n=1$, $I_1^{(k)}:=I^{(k)}$. It is easy to show that $I^{(k+1)}$ is an open, connected interval provided (i) $I^{(k)}$ is connected and (ii) $y_1 \notin I^{(k)}$.

Suppose $y_1 \in I^{(0)}$. Then we may assign $F(y_1) \equiv y_1 \in I^{(0)}$ so that F is monotone and admits a fixed point at y_1. Since the rotation number of F is independent of the choice of $F(y_1)$ (Section 4), this contradicts the irrationality of $\bar{\alpha}(F)$.

Similarly, for $k \geqslant 0$, if $y_1 \in I^{(k)}$, then we may assign a value $F(y_1) = \tilde{y}_1 \in I^{(0)}$ such that $F^k(y_1) = y_1$, which again contradicts the irrationality of $\bar{\alpha}(F)$. Thus, we proved that $I^{(k)}$ are connected intervals for $k \geqslant 0$.

In order to prove the mutual disjointedness of $I^{(k)}$, it is enough to show that F^k admits k distinct points of discontinuity. In fact,

$$\mathbb{T} \backslash F^k(\mathbb{T}) = \bigcup_{j=0}^{k} I^{(j)}$$

and the result follows by the strict monotonicity of F. The discontinuity points of F^k are given by

$$\{y_1\}, \{F^{-1}_{(y_1)}\}, ..., \{F^{-k}_{(y_1)}\}$$

$F^{-j}_{(y_1)}$ is well defined provided $F^{-j+1}_{(y_1)} \notin I^{(0)}$. Obviously, $y_1 \notin I^{(0)}$ and if $F^{-j}(y_1) \in I^{(0)}$, then

$$\{y_1\} \in F^j_{(I^{(0)})} = I^{(j)}$$

which contradicts the first part of the proof. Now assume $F^{-j}_{(y_1)} = F^{-l}_{(y_1)}$ for $j \neq l$. Then $F^{-j-1}_{(y_1)}, ..., F^{-l}_{(y_1)}$ is a closed orbit of F and we obtain a contradiction to the irrationality of $\bar{\alpha}(F)$. If $n > 1$, then the same proof works for each $I_j^{(\cdot)}$, $1 \leqslant j \leqslant n$, yielding (a), (b). If for each $1 \leqslant j \leqslant n$ we can find $1 \leqslant l \leqslant n$ and $k \geqslant 0$ for which $\{y_l\} \in I_j^{(k)}$, then there must be a periodic orbit of F for some choice of $F(y_i) \in I_i^{(0)}$, $i = 1, ..., n$. This proves (c). ∎

Let the rotation number α be given as a continued-fraction expansion, $\alpha = [a_1, a_2, ...]$, $a_i \in \mathbb{N}$. Let $\{q_n\}$ be the Fibonachi series defined by

$$q_1 = 1, \qquad q_2 = \left[\frac{1}{\alpha}\right], \qquad q_{n+1} = a_{n+1} q_n + q_{n-1} \tag{6.5}$$

where $[\cdot]$ stands for the integer part. The geometric interpretation of q_n is as follows:

Consider the shift mapping $t \to t + \alpha$ on the unit circle and let $\{t_n\}_{n=0}^{\infty}$ be the orbit of $t_0 = 0$ ($t_n = \alpha n$ *mode* 1). Then

$$\text{dist}(t_{q_n}, 0) < \text{dist}(t_j. 0); \qquad 1 \leqslant j < q_n$$

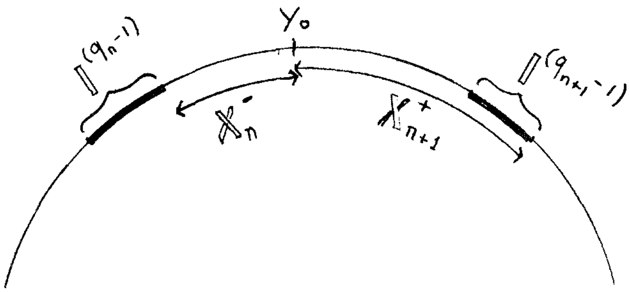

Fig. 6. Representation and ordering of the intervals $I_k^{(q_k-1)}$ for $k=n$, $n+1$, $n+2$, $n+3$. Here j is a given index for which x_j is a discontinuity point of $\tilde{F}$ (corresponding to one of the repelling resonance points). Here $x_j = 0$.

where dist$(\cdot)$ stands for the distance on the unit circle. Moreover, if $\alpha < 1/2$, then t_{q_n} is to the right of 0 if n is odd, and to the left of 0 if n is even. In particular,

$$0 < t_{q_{n+2}} < tq_n \qquad \text{if } n \text{ is odd} \tag{6.6}$$

$$t_{q_n} < t_{q_{n+2}} < 0 \qquad \text{if } n \text{ is even} \tag{6.7}$$

Pick up $j \in \{1,..., n\}$, and consider $I_j^{(q_n-1)}$, $n = 0, 1, 2,...$. By Lemmas 5.1 and 6.3, the ordering of $I_j^{(q_n-1)}$ with respect to the discontinuity point $\{y_j\}$ is the same ordering as the ordering of t_{q_n} with respect to $\{0\}$. [Notice that $I_j^{(n-1)}$ may be considered as the image $F^n_{(y_j)}$.] Let X_n^+ be the farthest distance of $I_j^{(q_n-1)}$ from $\{y_j\}$ and, likewise, let X_n^- be the least distance of $I_j^{(q_n-1)}$ from $\{y_j\}$ (see Fig. 6).

Then, (6.6) and (6.7) together with Lemma 6.3 yield

$$I_j^{(q_{n+2}-1)} \subset (0, X_{n+2}^+) \subset (0, X_n^-) \qquad \text{if } n \text{ is odd} \tag{6.8}$$

$$I_j^{(q_{n+2}-1)} \subset (-X_{n+2}^+, 0) \subset (-X_n^-, 0) \qquad \text{if } n \text{ is even} \tag{6.9}$$

Lemma 6.4. Set

$$\mathscr{K} := \oint_{C^\varepsilon} \log \frac{dF}{dy}\, dv \tag{6.10}$$

and[4]

$$\tau := \frac{\lambda_1 + \lambda_2}{\lambda_1}$$

[4] Notice that $1 > \tau > 0$ if the saddle's trace is negative.

where λ_i are the eigenvalues of the corresponding saddle, $\lambda_2 > 0 > \lambda_1$. Let y_j be the discontinuity point of F corresponding to either:

(a) The unique saddle point of negative trace, as in the assumption of Theorem 3.1.

(b) Or, if all traces are zero, let $j = m$ as in Lemma 6.3(c).

Then

$$\text{meas}(I_j^{(q_{n+1}-1)}) \geqslant C|X_n^-|^\tau e^{\mathscr{K} q_{n+1}} \tag{6.11}$$

where $C > 0$ is independent of n.

The proof of Lemma 6.4 is given in Appendix B.

Proof of Lemma 6.2. We can immediately exclude the case $\mathscr{K} > 0$ in (6.10). Assume $\mathscr{K} = 0$. If all traces of the saddle points are zero, then $\tau = 0$ and (6.11) yields an explicit estimate for $\text{meas}(I_j^{q_{2n}-1})$. In that case we obtain a uniform lower bound on $\text{meas}(I_j^{q_{2n}-1})$, contradicting their mutual disjointedness [Lemma 6.3(b)].

We now turn to the case $\tau > 0$ (negative trace) and assume, counterpositively, $\mathscr{K} = 0$. Let y_j be as in Lemma 6.4(a). Then

$$|X_{n+1}^+| \geqslant \text{meas}(I_j^{(q_{n+1}-1)}) \geqslant C|X_n^-|^\tau \geqslant C|X_{n+2}^+|^\tau$$

Hence

$$|X_{n+1}^+| \leqslant (C'|X_n^+|)^{1/\tau} \tag{6.12}$$

Thus, for $n \geqslant n_0$

$$\log|X_n^+| < \left(\frac{1}{\tau}\right)^{n-n_0} \log|X_{n_0}^+| + \left(\frac{1}{\tau}\right)^{n-n_0} \frac{1}{1-\tau} \log C'$$

Fix $D > 0$ and let n_0 be large enough such that

$$|X_{n_0}^+| < e^{-D}(C')^{1/(1-\tau)} \tag{6.13}$$

Hence

$$|\log(|X_n^+|)| \geqslant \left(\frac{1}{\tau}\right)^{n-n_0} \left[|\log(|X_{n_0}^+|)| + \frac{1}{\tau-1} \log C'\right] \geqslant D\left(\frac{1}{\tau}\right)^{n-n_0}$$

Let $\mathscr{U}_\delta(y_j) \subset C^\varepsilon$ be a small neighborhood of y_j. Then, applying Lemma 4.5,

$$\oint_{C^\varepsilon} \log\left(\frac{d}{dy}F\right) dv = \int_{\mathscr{U}_\delta(y_j)} \log\left(\frac{d}{dy}F\right) dv + \oint_{C^\varepsilon \setminus \mathscr{U}_\delta(y_j)} \log\left(\frac{d}{dy}F\right) dv$$

$$\leqslant \tau \int_{\mathscr{U}_\delta(y_j)} \log|y - y_j|\, dv + O(1) \tag{6.14}$$

We now estimate the integral on the right-hand side of (6.14) from above. For this sake, we restrict ourselves to the integral on a right neighborhood of $\{y_j\}$ defined by

$$\mathscr{U}_\delta^+ := (y_j, X_{2N+1}^+)$$

where N is sufficiently large so that $2N+1 > n_0$ and $\mathscr{U}_\delta^+ \subset \mathscr{U}_\delta(y_j)$. Then

$$\begin{aligned}\int_{\mathscr{U}_\delta^+} \log|y-y_j|\,dv &\leqslant \sum_{n=N}^{\infty} \log(|X_{2n+1}^+|)\,v(X_{2n+3}^+, X_{2n+1}^+)\\ &\leqslant -D\tau^{n_0} \sum_{n=N}^{\infty} \frac{1}{\tau^{2n+1}}\,v(X_{2n+3}^+, X_{2n+1}^+) \qquad (6.15)\end{aligned}$$

where the right-hand side of (6.15) is an upper Riemann sum for the integral on the left. By definition,

$$\begin{aligned}v(X_{2n+3}^+, X_{2n+1}^+) &= \operatorname{dist}(t_{q_{2n+3}}, t_{q_{2n+1}})\\ &= a_{2n+3}\operatorname{dist}(0, t_{q_{2n+2}}) := a_{2n+3}\,|t_{q_{2n+2}}|\end{aligned}$$

The second equality above follows since $(t_{q_{2n+3}}, t_{q_{2n+1}})$ contains exactly a_{2n+3} segments of length $|t_{q_{2n+2}}|$, where $\{a_i\}$ is as in (6.5). So the right-hand side of (6.15) is estimated by

$$\int_{\mathscr{U}_\delta^+} \log|y-y_j|\,dv \leqslant -D\tau^{n_0} \sum_{n=N}^{\infty} a_{2n+3}\tau^{-(2n+2)}\,|t_{q_{2n+2}}| \qquad (6.16)$$

On the other hand,

$$|t_{q_k}| < (q_{k+2}+1)^{-1}$$

holds $\forall k \in \mathbb{N}$. Indeed, the orbit $t_1, \ldots, t_{q_{k+2}}$ on the circle admits a maximal gap of $|t_{q_k}|$ between neighbor points. From (6.5):

$$q_k \leqslant \prod_{j=1}^{k} (a_j+1) \leqslant |\sup_{j\in\mathbb{N}}(a_j+1)|^k$$

so the sum on the right-hand side of (6.16) diverges provided

$$1/\tau \geqslant \sup_{j\in\mathbb{N}}\{a_j\}+1 \qquad (6.17)$$

Since, by definition,

$$0 < \tau = -\frac{2g_x(x_j, y_j)}{|\Omega(x_j)\,g_y(x_j, y_j)|^{1/2}}\,\varepsilon^{1/2} + O(\varepsilon) \qquad (6.18)$$

condition (6.17) is satisfied for a Liouville type number, if ε is small enough. This concludes the proof of Lemma 6.2. In particular, $\mathscr{K} = -\infty$ [see (6.10)], as we could expect in view of Theorem 5.1. ▌

APPENDIX A

Proof of Lemma 2.1. Let $(\bar{x}_1 \cdots \bar{x}_q)$ be the unstable critical points of (2.2), i.e.,

$$\frac{d}{dx} \bar{g}(x_i) > 0, \qquad i = 1,..., q$$

and let $\mathscr{U} \subset \mathbb{S}$ be a small neighborhood of $(\bar{x}_1,..., \bar{x}_q)$. Given $\delta > 0$, we can find $T > 0$ such that for any initial data $x(0) \in \mathbb{S} \backslash \mathscr{U}$, the corresponding orbit will be found at a $\delta/4$ neighborhood of the stable critical points $\bar{x}_{q+1},..., \bar{x}_k$ after $t \geqslant T/\varepsilon$. Now a trivial application of Neistadt's theorem[9] yields: There exists $\eta(\varepsilon) > 0$, $\lim_{\varepsilon \to 0} \eta(\varepsilon) = 0$, such that the x projection of the orbits of (2.1_ε) is approximated within the $\delta/4$ neighborhood of the orbits of (2.2) over a T/ε time interval, for all but a set N^ε of initial data where $\operatorname{meas}(N^\varepsilon) < \eta(\varepsilon)$. Thus, for $(x_0, y_0) \notin N^\varepsilon \cup (\mathscr{U} \oplus \mathbb{T})$, we conclude that $(x(t), y(t))$ will be found at a δ neighborhood of the stable critical points of (2.2_ε), i.e., $\{\bar{x}_i\} \oplus \mathbb{T}$, $i = q+1,..., k$, at time $t = T/\varepsilon$. It is easy to see that if δ is small enough, any orbit starting from a δ neighborhood of the above set will stay there indefinitely. Hence $\mathscr{D}(\{\mu\}_e^\varepsilon) \subseteq N^\varepsilon \cup (\mathscr{U} \oplus \mathbb{T})$. Since the Lebesgue measures of $\mathscr{U}$ and N^ε are arbitrarily small, the lemma is proved. ▌

APPENDIX B

Proof of Lemma 5.2. (a) $t(\cdot)$ is ν-integrable. Let $\mathscr{K}(\mathbb{T}^2)$ be the set of all points in $\mathbb{T}^2$ excluding the critical points of (2.1_ε) and their stable manifolds. Let $\{\mu\}_z^\varepsilon$ be the set of asymptotic invariant measures due to orbits starting at $\{z\}$ (see Section 1). We need to show the following:

(1) *Existence*: $\tilde{\mu} \in \{\mu\}_z^\varepsilon$, $\forall z \in \mathscr{K}(\mathbb{T}^2)$, and is singular with respect to all invariant measures supported on the critical points of (2.1_ε).

(2) *Uniqueness*: $\{\mu\}_z^\varepsilon$ is a singleton $\forall z \in \mathscr{K}(\mathbb{T}^2)$.

Proof of Existence. Let

$$\mathscr{K}(C^\varepsilon) := \mathscr{K}(\mathbb{T}^2) \cap C^\varepsilon$$

By definition, $\mathscr{K}(C^{\varepsilon})$ is invariant with respect to F and $\forall y \in \mathscr{K}(C^{\varepsilon})$

$$\{F^k(y) \notin \{y_1, ..., y_n\}, k = 0, 1, 2, ...\}$$

where $\{y_1, ..., y_n\}$ are the discontinuity points of F on C^{ε}.

By Definition 5.1, $t(y) < \infty$ if $y \notin \{y_1, ..., y_n\}$. Let

$$T_n^{(y)} = \sum_{j=0}^{n-1} t(F_{(y)}^j)$$

By the invariance of $\mathscr{K}(C^{\varepsilon})$ with respect to F, we conclude

$$T_n^{(y)} < \infty \quad \forall y \in \mathscr{K}(C^{\varepsilon}); \qquad n = 1, 2, ...$$

Let $\phi \in C^0(\mathbb{T}^2)$ be a test function. Then $P(\phi, \cdot) \in \mathbb{L}_1(dv)$ by the assumption of the lemma [cf. (5.3)]. Using the definition of $P(\phi, \cdot)$ [cf. (5.2)] and the Birkhoff ergodic theorem, we obtain

$$\oint_{C^{\varepsilon}} P(\phi, w)\, v(dw) = \lim_{n \to \infty} \frac{1}{n+1} \int_0^{T_n^{(y)}} \phi(z_{(s)}^{(y)})\, ds \tag{B.1}$$

a.s. $y \in C^{\varepsilon}$. Moreover, since v is uniquely ergodic of the circle map defined by F, (B.1) holds for *every* $y \in \mathscr{K}(C^3)$. In particular,

$$\bar{T} = \oint_{C^{\varepsilon}} P(1, w)\, v(dw) = \lim_{n \to \infty} \frac{1}{n+1} T_n^{(y)} < \infty$$

holds independently of $y \in \mathscr{K}(C^{\varepsilon})$. Thus,

$$\int_{\mathbb{T}^2} \phi\, d\tilde{\mu} := \frac{1}{\bar{T}} \oint_{C^{\varepsilon}} P(\phi, w)\, v(dw) = \lim_{n \to \infty} (T_n^{(y)})^{-1} \int_0^{T_n^{(y)}} \phi(z_{(s)}^{(y)})\, ds \tag{B.2}$$

$\forall y \in \mathscr{K}(C^{\varepsilon})$. This implies that $\tilde{\mu}$ is an invariant measure of (2.1_{ε}) and $\tilde{\mu} \in \{\mu\}_z^{\varepsilon}$, $\forall z \in \mathscr{K}(C^{\varepsilon})$. We now show that $\tilde{\mu}$ is singular with respect to the atomic measures supported on the critical points of (2.1_{ε}). Indeed, let $\{z_1, ..., z_k\}$ be the set of critical points of (2.1_{ε}) and $\mathscr{U}^{\delta}$ be a δ neighborhood of $\{z_1, ..., z_k\}$. Also let ϕ^{δ} be the indicator function on $\mathbb{T}^2 \backslash \mathscr{U}^{\delta}$. Obviously $P(\phi^{\delta}, \cdot)$ is uniformly bounded on C^{ε} (the bound depends, however, on δ), and

$$\lim_{\delta \to 0} \oint_{C^{\varepsilon}} P(\phi^{\delta}, y)\, v(dy) = \oint_{C^{\varepsilon}} P(1, y)\, v(dy) = \bar{T}$$

Hence, by (5.4),

$$\lim_{\delta \to 0} \tilde{\mu}(\mathbb{T}^2 \backslash \mathscr{U}^{\delta}) = 1 \Rightarrow \tilde{\mu}(\{z_i\}) = 0, \qquad i = 1, ..., k$$

Proof of Uniqueness. Let $y \in \mathscr{K}(C^\varepsilon)$ and assume, counterpositively, the existence of $\bar{\mu} \neq \tilde{\mu}$ and $\hat{T}_n \to \infty$ for which

$$\int_0^{\hat{T}_n} \phi(z_{(s)}^{(y)})\, ds = \hat{T}_n \int_{\mathbb{T}^2} \phi\, d\bar{\mu} + o(\hat{T}_n) \tag{B.3}$$

Let $\mathscr{U}^\delta$ be a δ neighborhood of the critical points of (2.1$_\varepsilon$), denoted by $\{z_1,..., z_k\}$. We consider two possibilities:

(i) There exists a subsequence $\hat{T}_{l(j)}$ for which

$$z^{(y)}(\hat{T}_{l(j)}) \notin \mathscr{U}^\delta \quad \forall k = 1, 2,... \tag{$*$}$$

(ii) There exist a subsequence $\hat{T}_{l(j)}$ and some $z_i \in \{z_1,..., z_k\}$ such that

$$z^{(y)}(\hat{T}_{l(j)}) \in B^\delta(z_i) \subset \mathscr{U}^\delta \tag{$**$}$$

where B^δ is a δ neighborhood of z_i.

Obviously, by excluding these two possibilities, we obtain a contradiction to (B.3).

(i) Let $\tau_{(z)}$ be the first intersection time of an orbit starting from $z \in \mathbb{T}^2$ with C^ε. Set τ^δ as the upper bound

$$\tau_{(z)} < \tau^\delta; \qquad \forall z \in \mathbb{T}^2 \backslash \mathscr{U}^\delta$$

Let

$$q_{(k)} = \min\{1 \leqslant i \leqslant k;\ T_i^{(y)} > \hat{T}_{j(k)}\} \tag{B.4}$$

By ($*$),

$$\hat{T}_{j(k)} < T_{q(k)}^{(y)} < \hat{T}_{j(k)} + \tau^\delta$$

Thus we get from (B.3)

$$\int_0^{T_{q(k)}^{(y)}} \phi(z_{(s)}^{(y)})\, ds = T_{q(k)}^{(y)} \int_{\mathbb{T}^2} \phi\, d\bar{\mu} + o(T_{q(k)}^{(y)} + O(\tau^\delta) \tag{B.5}$$

But, from the existence proof,

$$\lim_{n \to \infty} \frac{1}{T_n^{(y)}} \int_0^{T_n^{(y)}} \phi(z_{(s)}^{(y)})\, ds = \int_{\mathbb{T}^2} \phi\, d\tilde{\mu} \tag{B.6}$$

Since $\tilde{\mu} \neq \bar{\mu}$ by assumption, (B.5) and (B.6) contradict (B.3).

(ii) Let

$$\tau_{(z)}^{(i,\delta)} = \{\inf t;\ z(s) \notin B^\delta(z_i),\ z(0) := z \in B^\delta(z_i)\}$$

Then

$$\hat{T}_{l(k)} + \tau^{(i,\delta)}_{z(\hat{T}_{l(k)})} < T^{(y)}_{q(k)} < \hat{T}_{l(k)} + \tau^{(i,\delta)}_{z(\hat{T}_{l(k)})} + \tau^{\delta} \tag{B.7}$$

where $q(k)$ is defined as in (B.5) (this time with respect to $\hat{T}_{l(k)}$). Let $z_k(\cdot)$ be the solution of (2.1_ε) starting at $z_k(0) := z^{(y)}(\hat{T}_{l(k)})$. Recall from $(**)$ that $z^{(y)}(\hat{T}_{l(k)}) \in B^{\delta}(z_i)$. Utilizing (B.7), we obtain

$$\int_0^{T^{(y)}_{q(k)}} \phi(z^{(y)}_{(s)})\, ds = \int_0^{\hat{T}_{l(k)}} \phi(z^{(y)}_{(s)})\, ds + \int_0^{\hat{\tau}^{(i,\delta)}_{z_k(0)}} \phi(z_k(s))\, ds + O(\tau^{\delta}) \tag{B.8}$$

The second term on the right of (B.8) is approximated by

$$\tau^{i,\delta}_{z_k(0)} \{\phi(z_i) + O(\delta)\}$$

so

$$\int_0^{T^{(y)}_{q(k)}} \phi(z^{(y)}_{(s)})\, ds = \hat{T}_{l(k)} \int_{\mathbb{T}^2} \phi(z)\, \bar{\mu}(dz) + \tau^{i,\delta}_{z_k(0)} \{\phi(z_i) + O(\delta)\}$$
$$+ O(\hat{T}_{l(k)}) + O(\tau^{\delta}) \tag{B.9}$$

Passing to a subsequence, we can assume

$$\lim_{k\to\infty} \frac{\hat{T}_{l(k)}}{T^{(y)}_{q(k)}} = \beta, \qquad 0 \leqslant \beta \leqslant 1 \tag{B.10$_1$}$$

From (B.7) we obtain, for the same subsequence,

$$\lim_{k\to\infty} \frac{\tau_{z_k(0)}}{T^{(y)}_{q(k)}} = 1 - \beta \tag{B.10$_2$}$$

Substituting (B.10_1) and (B.10_2) in (B.9), we get

$$\int_0^{T^{(y)}_{q(k)}} \phi(z^{(y)}_{(s)})\, ds = T^{(y)}_{q(k)} \left\{\beta \int_{\mathbb{T}^2} \phi(z)\, \bar{\mu}(dz) + (1-\beta)\, \phi(z_i)\right\}$$
$$+ O(\delta T^{(y)}_{q(k)}) + O(\tau^{\delta}) + o(T^{(y)}_{q(k)}) \tag{B.11}$$

Since δ os arbitrarily small, (B.5) and (B.11) yield

$$\tilde{\mu} = \beta\bar{\mu} + (1-\beta)\, \delta(z_i)$$

But $\tilde{\mu}$ is nonatomic at any of the critical points. Hence $\beta = 1$ and $\bar{\mu} = \tilde{\mu}$.

(b) $t(\cdot)$ is not ν-integrable. In this case

$$\lim_{n\to\infty} \frac{T^{(y)}_n}{n} = \infty \qquad \forall y \in \mathscr{K}(C^{\varepsilon})$$

If ϕ is a test function on $\mathbb{T}^2$, supported outside the neighborhood of the singular points of (2.1_ε), then evidently

$$\int_{C^\varepsilon} P(\phi, y)\, \nu(dy) < \infty$$

Hence, if $y \in \mathscr{K}(C^\varepsilon)$,

$$\lim_{n \to \infty} \frac{1}{Y_n^{(y)}} \int_0^{T_n^{(y)}} \phi(z_{(s)}^{(y)})\, ds = \lim_{n \to \infty} \frac{n}{T_n^{(y)}} \int_{C^\varepsilon} P(\phi, y)\, \nu(dy) = 0$$

Therefore, all limit measures in $\{\mu\}_z^\varepsilon$, $z \in \mathscr{K}(C^\varepsilon)$, are supported on the critical points of (2.1_ε). ∎

APPENDIX C

Proof of Lemma 6.4. The proof is based on an extension of the well-known Denjoy inequality, which we quote below.

Let p/q be a rational approximation of α, satisfying

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \tag{C.1}$$

Then, for every $\phi \in BV(\mathbb{T})$,

$$\left|\sum_{j=0}^{q-1} \phi(F^j(y)) - q \oint_{\mathbb{T}^1} \phi\, d\nu\right| \leqslant \operatorname*{Var}_{\mathbb{T}^1}(\phi) \tag{C2}$$

where $F(\cdot)$ stands for a circle map with the rotation number α.

For the proof of (C.2) see, e.g., ref. 1, Chapter 3.11. Here we consider the case where ϕ is of bounded variation on $\mathbb{T} \backslash U_\delta(0)$, where $U_\delta(0)$ is a small neighborhood of $\{0\}$. Assume

$$\lim_{y \to 0} \phi(y) = -\infty$$

$\phi < 0$ and monotone on the right and left neighborhoods of 0 in $U_\delta(0)$, respectively. We claim:

Let $z_0 \in \mathbb{T}$, and q as in (C.1). Let $0 \leqslant k$, $l \leqslant q-1$ be such that $z_R := F^k(z_0)$ is the closest point to $\{0\}$ on the orbit $z_0, \ldots, F^{q-1}(z_0)$ from the right, and similarly $z_L := F^l(z_0)$ is the closest point to $\{0\}$ on the same orbit from the left. Let $J = (z_L, z_R) \supset \{0\}$. Then

$$\sum_{j=0}^{q-1} \phi(F^j(z_0)) \geqslant q \oint_{\mathbb{T}^1} \phi\, d\nu - \operatorname*{Var}_{\mathbb{T}^1 \backslash J}(\phi) \tag{C.3}$$

The proof of (C.3) follows directly from the proof of Denjoy's lemma, and we skip it. Choosing q large enough, so $J \subseteq U_\delta\{0\}$, we have

$$\operatorname*{Var}_{\mathbb{T}^1\setminus J}(\phi) \geqslant \operatorname*{Var}_{\mathbb{T}^1\setminus U_\delta\{0\}}(\phi) - [\phi(z_R)+\phi(z_L)]$$

[Notice that $\phi(z_R)+\phi(z_L)<0$.] So, under the same conditions,

$$\sum_{j=0}^{q-1} \phi(F^j(z_0)) \geqslant q \oint_{\mathbb{T}^1} \phi\, dv - \operatorname*{Var}_{\mathbb{T}^1\setminus U_\delta(0)}(\phi) + \phi(z_R)+\phi(z_L) \tag{C.4}$$

Now, consider the conditions of Theorem 3.1. Let $\{y_j\}=\{0\}$ be the discontinuity point of F corresponding to the unique saddle point of (2.1_ε) which is of negative trace. From Lemma 4.5 we see that $\log((d/dy)F)$ can be substituted for ϕ in the local neighborhood $U_\delta\{0\}$.

Let $z_0 \in I_j^{(0)}$ and $q=q_n$. Assuming n is odd, then $z_R = F_{(z_0)}^{q_n-1}$ and $zL = F_{(z_0)}^{q_{n-1}-1}$. Using (C.4), we get

$$\begin{aligned}\log\left(\frac{d}{dy} F_{(z_0)}^{q_n}\right) &= \sum_{j=0}^{q_n-1} \log\left[\frac{d}{dy} F^j(z_0)\right]\\ &\geqslant q_n \mathscr{K} - \operatorname*{Var}_{\mathbb{T}^1\setminus U_\delta(0)}\left[\log\left(\frac{d}{dy}F\right)\right]\\ &\quad + \log\left[\frac{d}{dy}F(z_R)\right] + \log\left[\frac{d}{dy}F(z_L)\right]\end{aligned}$$

Thus

$$\begin{aligned}\log\frac{d}{dy}\,[F^{q_n-1}(z_0)] &= \log\left[\frac{d}{dy}F^{q_n}(z_0)\right] - \log\left[\frac{d}{dy}F(z_R)\right]\\ &\geqslant q_n\mathscr{K} + \log\left[\frac{d}{dy}F(z_L)\right] - \operatorname*{Var}_{\mathbb{T}^1\setminus U_\delta(0)}\left(\log\frac{d}{dy}F\right)\\ &\Rightarrow \frac{d}{dy}F^{q_n-1}(z_0) \geqslant \tilde{C}^{-1}\frac{d}{dy}F(z_L)e^{q_n\mathscr{K}}\end{aligned} \tag{C.5}$$

where $\tilde{C}=\tilde{C}(\delta)$ is independent of n. The inequality (C.5) holds $\forall z_0 \in I_j^{(0)}$, provided z_L is the leftmost point in $F^{q_{n-1}-1}(I_j^{(0)}) := I_j^{q_{n-1}-1}$. Therefore, we may choose $z_L = -X_{n-1}^-$, obtaining

$$\inf_{I_j^{(0)}} \frac{d}{dy}F^{q_n-1} \geqslant C\frac{d}{dy}F(-X_{n-1}^-)e^{q_n\mathscr{K}} \geqslant C|X_{n-1}^-|^{(\lambda_1+\lambda_2)/\lambda_1}e^{q_n\mathscr{K}} \tag{C.6}$$

where we applied Lemma 4.5. Using

$$\operatorname{meas}[I_j^{(q_n-1)}] \geqslant I_j^{(0)} \inf_{I_j^{(0)}} \frac{d}{dy} F^{q_n-1}$$

we complete the proof of (6.11).

If case (b) of Lemma 6.4 holds, then

$$\inf_{\mathbb{T}^1} \left(\frac{d}{dy} F^{q_n} \right) > C$$

independently of n, and (6.11) follows with $\lambda_1 + \lambda_2 = 0$. ∎

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